Flow invariance for nonlinear partial differential delay equations

WOLFGANG M. RUESS
Duisburg-Essen University, Essen, Germany

The object of the talk are partial differential delay equations of the form

$$\text{(FDE)} \quad \begin{cases} \dot{u}(t) + Bu(t) \ni F(u_t), & t \geq 0 \\ u_{|_I} = \varphi \in \hat{E}. \end{cases}$$

Here, $B \subset X \times X$ is a (generally) nonlinear and multivalued $\omega-$accretive operator (differential expression) in a Banach space X, and, for given $I = [-r, 0]$, $r > 0$ (finite delay), or $I = \mathbf{R}^-$ (infinite delay), and $t \geq 0$, $u_t : I \to X$ is the history of u up to $t : u_t(s) = u(t+s)$, $s \in I$. Moreover, $\varphi : I \to X$ is a given initial history out of a space E of functions from I to X, and F is a given history-responsive operator with domain $\hat{E} \subset E$ and range in X.

Aside from a survey on the existence theory for (FDE), the main point of the talk is the solution [2] to the problem of a (sufficient) flow-invariance condition for specific subsets $\hat{E}$ of the initial history space E.
It extends the known conditions for the special cases of (1) $B = 0$ of ordinary delay equations, or (2) B linear and $-B$ generating a C_0-semigroup, or (3) B m-accretive and $I = \{0\}$ (no delay) to the general nonlinear and delay case of the above (FDE).

One of the basic systematic ingredients of the proof is a result by Michel [1] of – approximately – 60/2 years ago.

The results apply to diffusive population models with temporal or spatio-temporal averages over the past history, such as

$$\begin{cases} \dot{u}(t) - \Delta u(t) \ni au(t)\left[1 - bu(t) - \int_{-1}^{0} u(t + r(s))d\eta(s)\right], & t \geq 0 \\ u_{|(-R,0]} = \varphi \end{cases}$$

as well as corresponding models with the Laplacian being replaced by more general and nonlinear diffusion/absorption operators of the form

$$-div\ a(\cdot, grad\, u) + \tilde{\beta}(u); \quad -a(\cdot, grad\, u) \cdot n \in \beta(u) \quad \text{on} \quad \partial\Omega$$

with $\beta \subset \mathbf{R} \times \mathbf{R}$ a maximal monotone graph.

[1] Michel Pierre, *Invariant closed subsets for nonlinear semigroups*, Nonlinear Analysis TMA 2 (1978), 107-117

[2] W.M. Ruess, *Flow invariance for nonlinear partial differential delay equations*, Trans. Amer. Math. Soc. 361 (2009), 4367-4403