ALMOST GLOBAL SOLUTIONS OF EIKONAL EQUATIONS AND DISTANCE FUNCTIONS

Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is continuous, differentiable at each point of a dense open subset $\mathbb{R}^n \setminus \mathcal{S}$ of $\mathbb{R}^n$ and satisfies

$$\|Du(x)\| = 1 \quad \text{for} \quad x \in \mathbb{R}^n \setminus \mathcal{S}.$$

at x. Here $\|\cdot\|$ is a norm on $\mathbb{R}^n$. The closed set $\mathcal{S}$ consists of potential singularities of u where $\|Du(x)\| = 1$ might fail to hold; for example, u might not be differentiable at a point of $\mathcal{S}$.

Suppose $\|\cdot\|$ is the Euclidean norm. Then for each unit vector p and $a \in \mathbb{R}$, $u(x) = a + \langle x, p\rangle$ satisfies $\|Du\| = \|p\| = 1$ everywhere. If $z \in \mathbb{R}^n$ and $a \in \mathbb{R}$, then

$$u(x) = a \pm \|x - z\|$$

satisfies our conditions with $\mathcal{S} = \{z\}$. We call these functions "cone functions". If u is the distance to a closed bounded line segment L, then it satisfies our assumptions with $\mathcal{S} = L$. The question arises if there are any cases in which the set of genuine singularities is larger than a singleton and smaller than a segment. The answer, in the Euclidean case, is "no"; if the Hausdroff 1-measure of $\mathcal{S}$ is 0, then u is either affine or a cone function.

With suitable cone functions, this result is true in greater generality, and, in particular, it holds for all the p-norms

$$\|x\| = \|x\|_p = \left(|x_1|^p + \cdots + |x_n|^p\right)^{1/p}, 1 < p < \infty.$$

The proof hinges on establishing the "distance formula"

$$u(x) = r + \operatorname{dist}(x, L_r) \quad \text{if} \quad u(x) \geq r,$$
$$u(x) = r - \operatorname{dist}(x, L_r) \quad \text{if} \quad u(x) \leq r,$$

where $L_r = \{x : u(x) = r\}$ is a nonempty level set of u and $\operatorname{dist}(x, L_r)$ is the distance from x to L_r measured in the norm $\|\cdot\|_*$ dual to $\|\cdot\|$. To establish the distance formula requires a study of properties of a certain mapping defined at intermediate points of "rays" of the distance function. This seems to be a new chapter in the geometry of finite dimensional normed spaces. The results in this regard for the cases $p \neq 2$ of the p-norms are perhaps less than obvious.

This is joint work with **Luis Caffarelli**.